NOTE ON CERTAIN ITERATED AND MULTIPLE INTEGRALS

By Wallie Abraham Hurwitz

Dirichlet* stated and used the following formula for inversion of the order of integration:

$$\int_0^a dx \int_0^x \phi(x, y)\, dy = \int_0^a dy \int_y^a \phi(x, y)\, dx.$$

Volterra† makes use of a similar formula, which he calls "Dirichlet's principle." The truth of the theorem is obvious when the integrand is continuous within and on the boundary of the field of integration. The object of the present paper is to justify the theorem when the integrand is allowed to become infinite in certain ways on the boundary of the field; and to extend it to space of n dimensions. Part II, which contains the generalisation to n dimensions, is entirely independent of Part I, and includes its results as a special case; the method of treatment is, however, slightly different.

While the results here stated, at least in so far as they relate to the case of two dimensions, may be deduced from a general theorem of de la Vallée Poussin,‡ the simple character of the reasoning here used seems to justify an independent treatment.

I.

A form of statement slightly different from that of Dirichlet and Volterra will be used, in which the integrals appear as the generalisation of integrals mentioned by Schlömilch.§

Theorem. *Let $f(x, y)$ be continuous and limited in the region*

$$R:\quad 0 < x, \quad 0 < y, \quad x + y < 1;$$

* *Crelle's Journal*, vol. 17 (1837), p. 45.

† Cf., for example, *Annali di Matematica*, vol. 25 (1897), p. 142.

‡ *Cours d'analyse infinitésimale*, vol. 2 (1906), §§ 73, 77.

§ *Analytische Studien*, vol. 1 (1848), §19. Dirichlet, Liouville and other writers had also discussed special cases of the integral here considered. Cf. the concluding paragraph of this paper.

and let $0 < \lambda,\ \mu,\ \nu \leqq 1$; *then*

$$\int_0^1 dx \int_0^{1-x} x^{\lambda-1}\, y^{\mu-1}(1-x-y)^{\nu-1} f(x,\, y)\, dy$$

$$= \int_0^1 dy \int_0^{1-y} x^{\lambda-1}\, y^{\mu-1}(1-x-y)^{\nu-1} f(x,\, y)\, dx. \qquad (1)$$

Let $|f(x, y)| \leqq M$. The integral

$$I = \int_0^{1-x} x^{\lambda-1}\, y^{\mu-1}\, (1-x-y)^{\nu-1} f(x,\, y)\, dy$$

is equivalent, by means of the change of variable $y = (1-x)s$, to

$$x^{\lambda-1}(1-x)^{\mu+\nu-1} \int_0^1 s^{\mu-1}\, (1-s)^{\nu-1} f[x,\, (1-x)s]\, ds.$$

The expression $x^{\lambda-1}\,(1-x)^{\mu+\nu-1}$, which now stands outside the integral sign, is continuous for $0 < x < 1$; the integral is readily seen to be uniformly convergent for $0 \leqq x \leqq 1$, and hence to define a continuous function, whose absolute value is less than $MB(\mu, \nu)$. Thus the integral I represents a continuous function for $0 < x < 1$; and

$$|I| \leqq MB(\mu,\, \nu)\; x^{\lambda-1}\,(1-x)^{\mu+\nu-1}.$$

Furthermore, I may be integrated between the limits 0 and 1; for if we substitute for it the above expression greater than $|I|$, we obtain the convergent integral

$$MB(\mu,\, \nu) \int_0^1 x^{\lambda-1}\, (1-x)^{\mu+\nu-1}\, dx.$$

Hence the iterated integral on the left of (1) is convergent. Since the two sides of (1) differ, essentially, only in having x and y interchanged, it follows that the iterated integral on the right of (1) also converges. It is evident that if we integrate over any intervals not reaching outside the intervals appearing in the integrals just considered, we shall still have convergent integrals.

The field of integration is here a triangle bounded by the coordinate axes and the line $x + y = 1$. Draw lines inside this triangle, near its sides and parallel to them:

$$x = \delta, \quad y = \delta, \quad x + y = 1 - \delta.$$

Within and on the boundary of the smaller triangle thus formed the integrand is continuous; integration may therefore be performed in either order with the same result:

$$\int_\delta^{1-2\delta} dx \int_\delta^{1-x-\delta} x^{\lambda-1} y^{\mu-1}(1-x-y)^{\nu-1} f(x, y)\,dy$$
$$= \int_\delta^{1-2\delta} dy \int_\delta^{1-y-\delta} x^{\lambda-1} y^{\mu-1}(1-x-y)^{\nu-1} f(x, y)\,dx. \quad (2)$$

We may write

$$\int_\delta^{1-2\delta} dx \int_\delta^{1-x-\delta} x^{\lambda-1}\ y^{\mu-1}(1-x-y)^{\nu-1} f(x, y)\,dy$$
$$= \int_\delta^{1-2\delta} dx \int_0^{1-x} x^{\lambda-1} y^{\mu-1}(1-x-y)^{\nu-1} f(x, y)\,dy - I_1 - I_2, \quad (3)$$

where

$$I_1 = \int_\delta^{1-2\delta} dx \int_0^{\delta} x^{\lambda-1} y^{\mu-1}(1-x-y)^{\nu-1} f(x, y)\,dy$$

and

$$I_2 = \int_\delta^{1-2\delta} dx \int_{1-x-\delta}^{1-x} x^{\lambda-1} y^{\mu-1}(1-x-y)^{\nu-1} f(x, y)\,dy.$$

When $y < \delta$, $(1-x-y)^{\nu-1} \leqq (1-x-\delta)^{\nu-1}$; hence

$$\left|\int_0^{\delta} x^{\lambda-1} y^{\mu-1}(1-x-y)^{\nu-1} f(x, y)\,dy\right| \leqq Mx^{\lambda-1}(1-x-\delta)^{\nu-1}\int_0^{\delta} y^{\mu-1}\,dy$$
$$= \frac{M}{\mu}\delta^{\mu} x^{\lambda-1}(1-x-\delta)^{\nu-1};$$

therefore $|I_1| \leqq \displaystyle\int_\delta^{1-2\delta} dx \left| \int_0^{\delta} x^{\lambda-1} y^{\mu-1}(1-x-y)^{\nu-1} f(x, y)\,dy \right|$

$$\leqq \frac{M}{\mu}\delta^{\mu} \int_\delta^{1-2\delta} x^{\lambda-1}(1-x-\delta)^{\nu-1}\,dx$$
$$\leqq \frac{M}{\mu}\delta^{\mu} \int_0^{1-\delta} x^{\lambda-1}(1-x-\delta)^{\nu-1}\,dx$$
$$= \frac{M}{\mu} B(\lambda, \mu)\,\delta^{\mu}(1-\delta)^{\lambda+\nu-1}. \quad (4)$$

Again, when $y > 1 - x - \delta$, $y^{\mu-1} \leqq (1 - x - \delta)^{\mu-1}$; hence

$$\left| \int_{1-x-\delta}^{1-x} x^{\lambda-1} y^{\mu-1}(1 - x - y)^{\nu-1} f(x, y)\, dy \right|$$
$$\leqq Mx^{\lambda-1}(1 - x - \delta)^{\mu-1} \int_{1-x-\delta}^{1-x} (1 - x - y)^{\nu-1}\, dy$$
$$= \frac{M}{\nu} \delta^{\nu} x^{\lambda-1} (1 - x - \delta)^{\mu-1};$$

therefore $|I_2| \leqq \int_{\delta}^{1-2\delta} dx \left| \int_{1-x-\delta}^{1-x} x^{\lambda-1} y^{\mu-1}(1 - x - y)^{\nu-1} f(x, y)\, dy \right|$

$$\leqq \frac{M}{\nu} \delta^{\nu} \int_{\delta}^{1-2\delta} x^{\lambda-1}(1 - x - \delta)^{\mu-1}\, dx$$
$$\leqq \frac{M}{\nu} \delta^{\nu} \int_{0}^{1-\delta} x^{\lambda-1}(1 - x - \delta)^{\mu-1}\, dx$$
$$= \frac{M}{\nu} B(\lambda, \nu)\, \delta^{\nu} (1 - \delta)^{\lambda+\mu-1}. \tag{5}$$

From (4) and (5) we see that

$$\lim_{\delta=0} I_1 = 0, \qquad \lim_{\delta=0} I_2 = 0;$$

and hence from (3) that the limit of the left-hand side of (2) is

$$\lim_{\delta=0} \int_{\delta}^{1-2\delta} dx \int_{0}^{1-x} x^{\lambda-1} y^{\mu-1}(1 - x - y)^{\nu-1} f(x, y)\, dy$$
$$= \int_{0}^{1} dx \int_{0}^{1-x} x^{\lambda-1} y^{\mu-1}(1 - x - y)^{\nu-1} f(x, y)\, dy. \tag{6}$$

The two sides of (2) are obtained from each other by exchanging the roles of x and y; it follows that also the limit of the right-hand side of (2) is

$$\int_{0}^{1} dy \int_{0}^{1-y} x^{\lambda-1} y^{\mu-1} (1 - x - y)^{\nu-1} f(x, y)\, dx. \tag{7}$$

If we now allow δ to approach zero in (2), making use of the facts stated in (6) and (7), we obtain the theorem to be proved.*

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* It is in fact true that the *double* integral over the region R converges and is equal to the iterated integrals. The proof of this theorem, which would require separate treatment according as $f(x, y)$ does or does not retain the same sign throughout R, is a special case of the more general considerations of Part II.

The form in which the theorem is used by Dirichlet, and by Volterra and other writers on integral equations, is obtainable from this result by means of the substitution $x' = b - (b-a)x$. $y' = a + (b-a)y$.
We find thus the

THEOREM. *Let $\psi(x, y)$ be continuous and limited within the triangle bounded by the lines $x = y$, $x = b$, $y = a$; and let $0 < \lambda, \mu, \nu \leqq 1$; then*

$$\int_a^b dx \int_a^x (b-x)^{\lambda-1}(y-a)^{\mu-1}(x-y)^{\nu-1}\psi(x, y)dy$$
$$= \int_a^b dy \int_y^b (b-x)^{\lambda-1}(y-a)^{\mu-1}(x-y)^{\nu-1}\psi(x, y)dx.$$

The preceding theorems may be stated for a wider class of functions. The equality of the results of iterated integration in the two orders in the interior triangle is sufficient for the validity of the proof as given.

II.

The preceding considerations are readily extended to the case of n variables. We have first the following

THEOREM. *Consider the n-dimensional region*

$$R: \begin{cases} 0 < x_i \; [i = 1, 2, \cdots n]; \\ x_1 + x_2 + \cdots + x_n < 1; \end{cases}$$

and the $(n-1)$-dimensional region

$$\bar{R}: \begin{cases} 0 < x_i \; [i = 1, 2, \cdots n-1]; \\ x_1 + x_2 + \cdots + x_{n-1} < 1. \end{cases}$$

Let $f(x_1, x_2, \cdots x_n)$ be continuous and limited in R, and let

$$0 < \lambda_1, \lambda_2, \cdots \lambda_n, \lambda \leqq 1;$$

then

$$\int_R x_1^{\lambda_1-1} x_2^{\lambda_2-1} \cdots x_n^{\lambda_n-1}(1 - x_1 - x_2 - \cdots - x_n)^{\lambda-1} f dS$$
$$= \int_{\bar{R}} d\bar{S} \int_0^{1-x_1-x_2-\cdots-x_{n-1}} x_1^{\lambda_1-1} x_2^{\lambda_2-1} \cdots x_n^{\lambda_n-1}(1 - x_1 - x_2 - \cdots - x_n)^{\lambda-1} f dx_n.$$

It should first be noticed that when $f(x_1, x_2, \cdots x_n) = 1$, the n-fold

integral is known to be convergent, and its value is

$$\frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\cdots\Gamma(\lambda_n)\Gamma(\lambda)^*}{\Gamma(\lambda_1+\lambda_2+\cdots+\lambda_n+\lambda)}.$$

It will be convenient to write

$$x_1^{\lambda_1-1}\, x_2^{\lambda_2-1}\cdots x_n^{\lambda_n-1}(1-x_1-x_2-\cdots-x_n)^{\lambda-1}=\phi(x_1,\,x_2,\cdots x_n).$$

Let $|f|\leqq M$, and suppose at first that $f\geqq 0$. The integral

$$I=\int_0^{1-x_1-\cdots-x_{n-1}}\phi\,dx_n,$$

subjected to the change of variable $x_n=(1-x_1-\cdots-x_{n-1})s$, becomes

$$x_1^{\lambda_1-1}\cdots x_{n-1}^{\lambda_{n-1}-1}(1-x_1-\cdots-x_{n-1})^{\lambda+\lambda_n-1}$$
$$\int_0^1 s^{\lambda_n-1}(1-s)^{\lambda-1}f[x_1,\cdots x_{n-1},(1-x_1-\cdots-x_{n-1})s]ds.$$

The integral which appears in the last expression converges uniformly throughout the region formed by $\bar{R}$ and its boundary, and therefore represents a continuous function in this region; the function defined by I is therefore of the same character in $\bar{R}$ as the given function ϕ in R.† In $\bar{R}$, I is continuous, and

$$|I|\leqq MB(\lambda,\lambda_n)\,x_1^{\lambda_1-1}\cdots x_{n-1}^{\lambda_{n-1}-1}\,(1-x_1-\cdots-x_{n-1})^{\lambda+\lambda_n-1}$$

Also we may integrate I over $\bar{R}$, since the integral

$$MB(\lambda,\lambda_n)\int_{\bar{R}}x_1^{\lambda_1-1}\cdots x_{n-1}^{\lambda_{n-1}-1}(1-x_1-\cdots-x_{n-1})^{\lambda+\lambda_n-1}d\bar{S}$$

(which belongs to the special type just mentioned) is convergent.

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* Goursat, *Cours d'analyse*, vol. I, §150. The proof there given under the hypothesis that the exponents are positive (so that the integral is *proper*) holds without alteration if each exponent is greater than -1.

† If $\lambda+\lambda_n\leqq 1$, we take the function defined by the above integral with respect to s as the function corresponding to f, and $\lambda+\lambda_n$ as the number corresponding to λ; if $\lambda+\lambda_n>1$, we take the integral multiplied by $(1-x_1-\cdots-x_{n-1})^{\lambda+\lambda_n-1}$ as the function corresponding to f, and 1 as the number corresponding to λ.

Consider now the smaller fields of integration:

$$R_\delta:\begin{cases}\delta < x_i \quad [i = 1, 2, \cdots n];\\ x_1 + x_2 + \cdots + x_n < 1 - \delta;\end{cases}$$

$$\bar{R}_\delta:\begin{cases}\delta < x_i \quad [i = 1, 2, \cdots n-1];\\ x_1 + x_2 + \cdots + x_{n-1} < 1 - 2\delta;\end{cases}$$

$$\bar{R}':\begin{cases}0 < x_i \quad [i = 1, 2, \cdots n-1];\\ x_1 + x_2 + \cdots + x_{n-1} < 1 - \delta.\end{cases}$$

Since the integrand is continuous within and on the boundary of R_δ,

$$\int_{R_\delta}\phi dS = \int_{\bar{R}_\delta} d\bar{S}\int_\delta^{1-x_1-\cdots-x_{n-1}-\delta}\phi dx_n. \tag{8}$$

As before, write

$$\int_{\bar{R}_\delta} d\bar{S}\int_\delta^{1-x_1-\cdots-x_{n-1}-\delta}\phi dx_n = \int_{\bar{R}_\delta} d\bar{S}\int_0^{1-x_1-\cdots-x_{n-1}}\phi dx_n - I_1 - I_2, \tag{9}$$

where

$$I_1 = \int_{\bar{R}_\delta} d\bar{S}\int_0^\delta \phi dx_n,$$

and

$$I_2 = \int_{\bar{R}_\delta} d\bar{S}\int_{1-x_1-\cdots-x_{n-1}-\delta}^{1-x^1-\cdots-x_{n-}}\phi dx_n.$$

When $x_n < \delta$, $(1-x_1-\cdots-x_n)^{\lambda-1} \leqq (1-x_1-\cdots-x_{n-1}-\delta)^{\lambda-1}$; hence

$$\int_0^\delta \phi dx_n \leqq M x_1^{\lambda_1-1}\cdots x_{n-1}^{\lambda_{n-1}-1}(1-x_1-\cdots-x_{n-1}-\delta)^{\lambda-1}\int_0^\delta x_n^{\lambda_n-1}dx_n$$

$$= \frac{M}{\lambda_n}\delta^{\lambda_n} x_1^{\lambda_1-1}\cdots x_{n-1}^{\lambda_{n-1}-1}(1-x_1-\cdots-x_{n-1}-\delta)^{\lambda-1};$$

therefore

$$I_1 \leqq \frac{M}{\lambda_n} \delta^{\lambda_n} \int_{\bar{R}_\delta} x_1^{\lambda_1 - 1} \cdots x_{n-1}^{\lambda_{n-1}-1}(1 - x_1 - \cdots - x_{n-1} - \delta)^{\lambda - 1} d\bar{S}$$

$$\leqq \frac{M}{\lambda_n} \delta^{\lambda_n} \int_{\bar{R}'} x_1^{\lambda_1 - 1} \cdots x_{n-1}^{\lambda_{n-1}-1}(1 - x_1 - \cdots - x_{n-1} - \delta)^{\lambda - 1} d\bar{S}^*$$

$$= \frac{M}{\lambda_n} \frac{\Gamma(\lambda_1) \cdots \Gamma(\lambda_{n-1})\Gamma(\lambda)}{\Gamma(\lambda_1 + \cdots + \lambda_{n-1} + \lambda)} \delta^{\lambda_n} (1 - \delta)^{\lambda_1 + \cdots + \lambda_{n-1} + \lambda - 1} \tag{10}$$

Also, when

$$x_n > (1 - x_1 - \cdots - x_{n-1} - \delta), \; x_n^{\lambda_n - 1} \leqq (1 - x_1 - \cdots - x_{n-1} - \delta)^{\lambda_n - 1};$$

hence $\displaystyle\int_{1 - x_1 - \cdots - x_{n-1} - \delta}^{1 - x_1 - \cdots - x_{n-1}} \phi dx_n$

$$\leqq M x_1^{\lambda_1 - 1} \cdots x_{n-1}^{\lambda_{n-1}-1}(1 - x_1 - \cdots - x_{n-1} - \delta)^{\lambda_n - 1} \int_{1 - x_1 - \cdots - x_{n-1} - \delta}^{1 - x_1 - \cdots - x_{n-1}} (1 - x_1 - \cdots - x_n)^{\lambda - 1} dx_n$$

$$= \frac{M}{\lambda} \delta^{\lambda} x_1^{\lambda_1 - 1} \cdots x_{n-1}^{\lambda_{n-1}-1}(1 - x_1 - \cdots - x_{n-1} - \delta)^{\lambda_n - 1};$$

therefore

$$I_2 \leqq \frac{M}{\lambda} \delta^{\lambda} \int_{\bar{R}_\delta} x_1^{\lambda_1 - 1} \cdots x_{n-1}^{\lambda_{n-1}-1}(1 - x_1 - \cdots - x_{n-1} - \delta)^{\lambda_n - 1} d\bar{S}$$

$$\leqq \frac{M}{\lambda} \delta^{\lambda} \int_{\bar{R}'} x_1^{\lambda_1 - 1} \cdots x_{n-1}^{\lambda_{n-1}-1}(1 - x_1 - \cdots - x_{n-1} - \delta)^{\lambda_n - 1} d\bar{S}$$

$$= \frac{M}{\lambda} \frac{\Gamma(\lambda_1) \cdots \Gamma(\lambda_{n-1})\Gamma(\lambda_n)}{\Gamma(\lambda_1 + \cdots + \lambda_{n-1} + \lambda_n)} \delta^{\lambda} (1 - \delta)^{\lambda_1 + \cdots + \lambda_{n-1} + \lambda_n}. \tag{11}$$

From (10) and (11) we see that

$$\lim_{\delta = 0} I_1 = 0, \qquad \lim_{\delta = 0} I_2 = 0;$$

and therefore from (9) that the limit of the right-hand side of (8) is

$$\lim_{\delta = 0} \int_{\bar{R}_\delta} d\bar{S} \int_0^{1 - x_1 - \cdots - x_{n-1}} \phi dx_n = \int_{\bar{R}} d\bar{S} \int_0^{1 - x_1 - \cdots - x_{n-1}} \phi dx_n. \tag{12}$$

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* This integral is evaluated by means of the substitution $x_i = (1 - \delta) y_i, [i = 1, 2, \cdots n - 1]$ which reduces it to the special case already mentioned.

As for the left-hand side of (8), we note that since the integrand *retains the same sign throughout* R and since for a *single set of regions* R_δ whose limit is R,

$$\lim_{\delta=0}\int_{R_\delta}\phi ds$$

exists, therefore the integral over R converges and

$$\int_R \phi dS = \lim_{\delta=0}\int_{R_\delta}\phi dS.$$

Hence we conclude from (8) and (12) that

$$\int_R \phi dS = \int_{\bar{R}} d\bar{S}\int_0^{1-x_1-\cdots-x_{n-1}}\phi dx_n,$$

which was the theorem to be proved.

Suppose now that f changes sign in R. It will always be possible to express f as the difference of two functions which are nowhere negative, each of which is continuous and limited in R.* The theorem will hold for each of these functions, and hence for their difference f.

We have thus reduced an n-fold integral to an $(n-1)$-fold integral of a simple integral. In the course of the proof it appeared that the function of $n-1$ variables resulting from the simple integration has the same properties in $\bar{R}$ as the given function of n variables has in R. We may therefore repeat the process time after time, until we obtain an iterated integral in n variables. Furthermore, at each step the selection of the variable with respect to which the simple integration is performed is evidently only a matter of notation; we have therefore the

THEOREM. *Let* $f(x_1, x_2, \cdots x_n)$ *be continuous and limited in the region*

$$R:\begin{cases}0 < x_i \quad [i = 1, 2, \cdots n]\,; \\ x_1 + x_2 + \cdots + x_n < 1\,;\end{cases}$$

and let $0 < \lambda_1, \lambda_2 \cdots \lambda_n \leqq 1$; *then the n-fold integral*

$$\int_R x_1^{\lambda_1-1}x_2^{\lambda_2-1}\cdots x_n^{\lambda_n-1}(1-x_1-x_2-\cdots-x_n)^{\lambda-1}f dS$$

* For instance, by writing $f = M - (M - f)$, where M is a constant greater than any value of f in R.

converges, and may be evaluated by iterated integration in any order; for example,

$$\int_R x_1^{\lambda_1-1} x_2^{\lambda_2-1} \cdots x_n^{\lambda_n-1}(1 - x_1 - x_2 - \cdots - x_n)^{\lambda-1} f\,dS$$
$$= \int_0^1 x_1^{\lambda_1-1}\,dx_1 \int_0^{1-x_1} x_2^{\lambda_2-1}\,dx_2 \cdots \int_0^{1-x_1-x_2-\cdots-x_{n-1}} x_n^{\lambda_n-1}(1 - x_1 - x_2 - \cdots - x_n)^{\lambda-1} f\,dx_n.$$

By a substitution of the form $x_i = \left(\frac{x_i'}{a_i}\right)^{\rho_i}$ and a slight change of notation, the theorem may be made to apply to the more general case in which the region of integration is

$$\begin{cases} 0 < x_i & [i = 1, 2, \cdots n]; \\ \left(\frac{x_1}{a_1}\right)^{\rho_1} + \left(\frac{x_2}{a_2}\right)^{\rho_2} + \cdots + \left(\frac{x_n}{a_n}\right)^{\rho_n} < 1; \end{cases}$$

and the integrand is

$$x_1^{\lambda_1-1} x_2^{\lambda_2-1} \cdots x_n^{\lambda_n-1}\left[1 - \left(\frac{x_1}{a_1}\right)^{\rho_1} - \left(\frac{x_2}{a_2}\right)^{\rho_2} - \cdots - \left(\frac{x_n}{a_n}\right)^{\rho_n}\right]^{\lambda-1} f(x_1, x_2, \cdots x_n),$$

where $0 < \lambda_i \leqq \rho_i$ $[i = 1, 2, \cdots n]$, $0 < \lambda \leqq 1$.

Special cases of integrals of the types discussed in this paper were considered by Dirichlet,* and shortly afterward by Liouville † and Catalan.‡ The most general case (belonging to this type) mentioned by these writers is obtained by taking for $f(x_1, x_2, \cdots x_n)$ in this paper a function of $x_1 + x_2 + \cdots + x_n$. The problem discussed is always the evaluation of the integral, or at least its reduction to a simple integral,—all necessary considerations as to convergence and related questions being assumed. All these results are given by Schlömilch.§ References to later work of like character will be found in *Encyclopädie der Mathematischen Wissenschaften*, II A3, footnotes 147–149.

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* *Abhandlungen der Akademie der Wissenschaften zu Berlin*, 1839, p. 61.

† *Journal de Mathématiques*, vol. 4 (1839), p. 225.

‡ *Journal de Mathématiques*, vol. 4 (1839), p. 323.

§ Loc. cit.